C12 Jan 2014 (MA)

Q1) $\left(2-\frac{x}{2}\right)^6 \simeq (2)^6 + \binom{6}{1}(2^5)\left(-\frac{x}{2}\right)^1 + \binom{6}{2}(2^4)\left(-\frac{x}{2}\right)^2.$

$\simeq 64 + 192\left(\frac{-x}{2}\right) + 240\left(\frac{x^2}{4}\right) + \ldots$

$\simeq 64 - 96x + 60x^2$

Q2a) $f(x) = \frac{8}{x^2} - 4x^{\frac{1}{2}} + 3x - 1$

$\therefore \quad = 8x^{-2} - 4x^{\frac{1}{2}} + 3x - 1$

$\therefore \; f'(x) = (-2)(8x^{-3}) - 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 3$

$= \frac{-16}{x^3} - \frac{2}{x^{\frac{1}{2}}} + 3$

b) $\int f(x)\,dx = \int 8x^{-2} - 4x^{\frac{1}{2}} + 3x - 1 \, dx$

$= \frac{8x^{-1}}{-1} - \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^2}{2} - x + c$

$= -\frac{8}{x} - \frac{8}{3}x^{3/2} + \frac{3}{2}x^2 - x + c$

3a) substituting $x = 2$ into $f(x)$:

$\leftarrow$ remainder

$$f(2) = 10(2^3) + 27(2^2) - 13(2) - 12 = \boxed{150}$$

ii) substituting $x = -3$

$$f(-3) = 10(-3)^3 + 27(-3)^2 - 13(-3) - 12 = \boxed{0}$$

$\therefore$ there is no remainder when $f(x)$ is divided by $(x+3)$.

b) from (ii), the remainder was 0 which means $(x+3)$ is a factor of $f(x)$.

using long division to obtain the 'quotient':

$$\begin{array}{r l}
 & 10x^2 - 3x - 4 \\
x+3 & \overline{)\,10x^3 + 27x^2 - 13x - 12} \\
(-) & \underline{10x^3 + 30x^2} \\
 & 0 - 3x^2 - 13x \\
(-) & \underline{-3x^2 - 9x} \\
 & 0 \quad -4x - 12 \\
 & \underline{-4x - 12} \\
 & 0 \quad 0
\end{array}$$

$$\therefore f(x) = (x+3)(10x^2 - 3x - 4)$$

but $(10x^2 - 3x - 4) = (5x - 4)(2x + 1)$

so $f(x) = \boxed{(x+3)(5x-4)(2x+1)}$

4i) rationalising the fraction . . . $(2\sqrt{2})^2 = 8.$

$$\frac{4}{2\sqrt{2}-\sqrt{6}} = \frac{4(2\sqrt{2}+\sqrt{6})}{(2\sqrt{2}-\sqrt{6})(2\sqrt{2}+\sqrt{6})}$$

$$= \frac{8\sqrt{2}+4\sqrt{6}}{8+\cancel{4\sqrt{3}}-\cancel{4\sqrt{3}}-6} = \frac{8\sqrt{2}+4\sqrt{6}}{2}$$

$$= 4\sqrt{2}+2\sqrt{6} = \underline{2\sqrt{2}(2+\sqrt{3})}$$

$\underline{\text{check}}$: $2\sqrt{2}(2+\sqrt{3}) = 4\sqrt{2} + 2\sqrt{2}\times\sqrt{3} = 4\sqrt{2}+2\sqrt{6}$ ✓

ii) $\text{LHS} = (\sqrt{27}) + (\sqrt{21}\times\sqrt{7}) - \frac{6}{\sqrt{3}}$

$$= \sqrt{9\times 3} + (\sqrt{7}\times\sqrt{7}\times\sqrt{3}) - \frac{6}{\sqrt{3}}$$

$$= 3\times\sqrt{3} + (7\times\sqrt{3}) - \frac{6\sqrt{3}}{3}$$

rationalising yields $\left(-\frac{6\sqrt{3}}{3}\right)$.

$$= 3\sqrt{3} + 7\sqrt{3} - 2\sqrt{3}$$

$$= (3+7-2)\sqrt{3} = \underline{8\sqrt{3}}$$

5a) $u_1 = 3$

$u_{n+1} = 2 - \frac{4}{u_n}$

$\therefore\ u_2 = 2 - \frac{4}{u_1} = 2 - \frac{4}{3} = \boxed{\frac{2}{3}}$

$u_3 = 2 - \frac{4}{u_2} = 2 - \frac{4}{\frac{2}{3}} = \boxed{-4}$

$u_4 = 2 - \frac{4}{u_3} = 2 - \frac{4}{-4} = \boxed{3}$

b) this sequence follows a pattern...

$\sum_1^n u_i = (3 + \frac{2}{3} - 4) + (3 + \frac{2}{3} - 4) + (3 + \frac{2}{3} - 4) + \ldots$

as $u_4 = 3$ and $u_1 = 3$, a "loop" is essentially formed.

every 3^{rd} term after u_1 is equal to 3

$\therefore\ \boxed{u_{61} = 3}$

c) $\sum_1^{99} u_i = (3 + \frac{2}{3} - 4) + (3 + \frac{2}{3} - 4) + \ldots$

$= 33 \times (3 + \frac{2}{3} - 4)$

$= 33 \times (-\frac{1}{3}) = \boxed{-11}$

6) $ab = 25$ — 1

$\log_4 a - \log_4 b = 3$ — 2

from 2 : $\log_4\left(\frac{a}{b}\right) = 3$ → (subtraction rule)

$\therefore 4^3 = \frac{a}{b} = 64$

$\therefore a = 64b$ — 3

Substituting 3 into 1 :

$(64b)(b) = 25$

$64b^2 = 25$

$b^2 = \frac{25}{64}$ $\therefore b = \boxed{\frac{5}{8}}$

and from 1, $a = \frac{25}{\frac{5}{8}} = \boxed{40}$

7a) $12\sin^2 x - \cos x - 11 = 0$

$\sin^2 x = 1 - \cos^2 x$

$\therefore 12(1-\cos^2 x) - \cos x - 11 = 0$

(Rearrange...)

$\Rightarrow 12\cos^2 x + \cos x - 1 = 0$

b) from (a), $12\cos^2 x + \cos x - 1 = 0$.

let $y = \cos x$,

$0 \leq x \leq 360°$

then $12y^2 + y - 1 = 0$

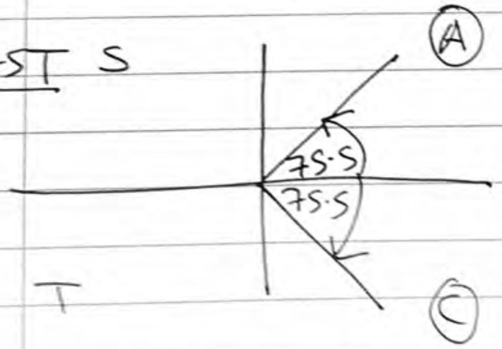
$\Rightarrow (4y-1)(3y+1) = 0$

$\therefore y = \frac{1}{4}$

$\cos x = \frac{1}{4}$

$x = \cos^{-1}\left(\frac{1}{4}\right) = 75.5°$

By CAST



$x = 75.5°, 360 - 75.5°$

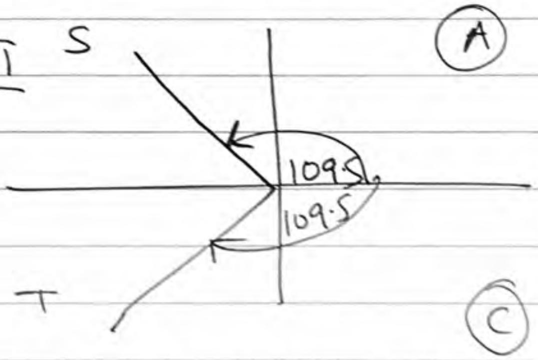
$\therefore x = \boxed{75.5°, 284.5°}$

$y = -\frac{1}{3}$

$\cos x = -\frac{1}{3}$

$x = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$

By CAST



$x = 109.5°, 360 - 109.5°$

$\boxed{x = 109.5°, 250.5°}$

8) $kx^2 + 8x + 2(k+7) = 0$

no real roots $\therefore$ $b^2 - 4ac < 0$

$a = k$
$b = 8$
$c = 2(k+7)$

$b^2 - 4ac < 0$

$(8)^2 - 4(k)(2)(k+7) < 0$

$64 - 8k(k+7) < 0$

$64 - 8k^2 - 56k < 0$

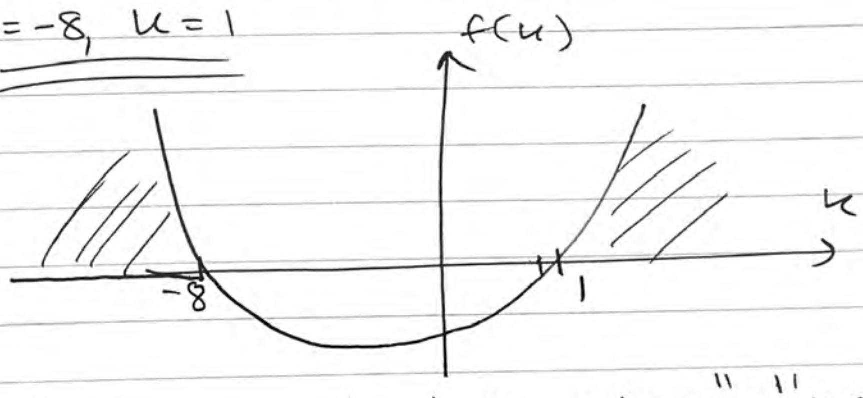
$\Rightarrow \quad 8k^2 + 56k - 64 > 0$

$\div 8: \quad k^2 + 7k - 8 > 0$

solving $[k^2 + 7k - 8 = 0]$ to find critical values:

$(k+8)(k-1) = 0$

$k = -8, \; k = 1$



Region required is where "y" > 0

hence values are: $\boxed{\begin{array}{l} k > 1 \\ k < -8 \end{array}}$

9a)

$a + ar + ar^2 + \ldots$

$300 \quad 300 \times 1.05 \quad \ldots$

Number sold in 24^{th} month $= 24^{th}$ term $= ar^{23}$

$= (300) \times (1.05)^{23} \simeq 921.5$

$\therefore$ $\boxed{921 \text{ phones sold}}$

b) $S_n = \dfrac{a(1-r^n)}{1-r}$.

$\therefore S_{24} = \dfrac{300(1-(1.05^{24}))}{1-1.05} = 13350.59966\ldots$

$\therefore$ $\boxed{13351}$ to the nearest integer

c) $300(1.05)^{n-1} > 3000$

number of phones sold in 'n^th' month.

$\Rightarrow (1.05)^{n-1} > 10$

$\Rightarrow \log((1.05)^{n-1}) > \log 10.$

$(n-1)\log(1.05) > \log(10)$

$\therefore n-1 > \dfrac{\log 10}{\log 1.05}$

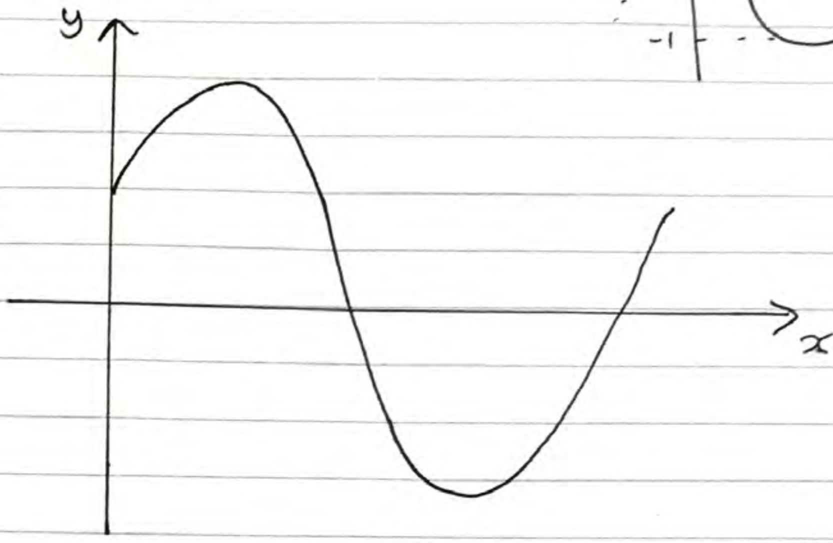
$n-1 > 47.194\ldots$

$\therefore n > 48.194\ldots$ so $\boxed{N = 49}$

(next integer)

10a) $y = \cos\left(x - \frac{\pi}{3}\right)$

TRANSLATION IN THE +VE x-direction BY $\left(\frac{\pi}{3}\right)$



b) at $y=0$: $\cos\left(x - \frac{\pi}{3}\right) = 0$

$$x - \frac{\pi}{3} = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2} = x - \frac{\pi}{3}$$

$$\text{so} \quad x = \frac{\pi}{2} + \frac{\pi}{3}, \frac{3\pi}{2} + \frac{\pi}{3}$$

$$x = \frac{5\pi}{6}, \frac{11\pi}{6}.$$

so $\boxed{\left(\frac{5\pi}{6}, 0\right)}$ and $\boxed{\left(\frac{11\pi}{6}, 0\right)}$

at $x=0$: $y = \cos\left(\frac{-\pi}{3}\right) = \frac{1}{2}$ $\therefore$ $\boxed{\left(0, \frac{1}{2}\right)}$

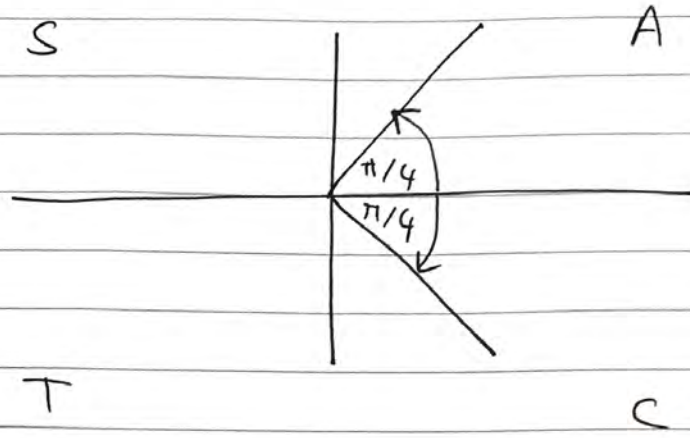
c) $0 \leq x \leq 2\pi$

$$\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x - \frac{\pi}{3} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

new interval : $-\frac{\pi}{3} \leq x-\frac{\pi}{3} \leq \frac{5\pi}{3}$

using CAST :



$x-\frac{\pi}{3}=\frac{\pi}{4}, -\frac{\pi}{4},$

$\therefore x = \frac{\pi}{4}+\frac{\pi}{3}, \quad -\frac{\pi}{4}+\frac{\pi}{3}$

$x = \boxed{\frac{7\pi}{12}}, \boxed{\frac{\pi}{12}}$

11a) $a \quad a+d \quad a+2d$

$60 \quad 4p \quad 2p-6$

$a+d=4p$ — ①

$a+2d=2p-6$ — ②

② − ① : $a+2d-a-d = 2p-6-4p$

$\therefore d = -2p-6$ //

substitute into ① : $a-2p-6 = 4p$

(a=60) $60-6 = 6p = 54$

$\therefore p = \frac{54}{6} = \boxed{9}$

b) n^{th} term $= a + (n-1)d$

20^{th} term $= 60 + 19d$

from (a), $d = -2p - 6 = -2(9) - 6 = \underline{\underline{-24}}$

$\therefore 20^{th}$ term $= 60 + 19(-24) = \boxed{-396}$

c) $S_n = \frac{n}{2}[2a + (n-1)d]$

$$\left.\begin{array}{l} d = -24 \\ a = 60 \end{array}\right\} S_n = \frac{n}{2}[120 + (n-1)(-24)]$$

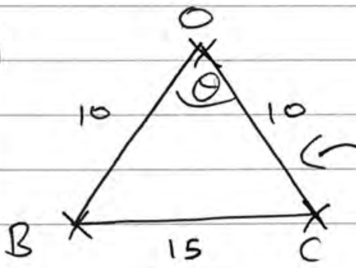
$$= \frac{n}{2}[120 + 24(1-n)]$$

compare with given result to obtain required expression:

$$S_n = \frac{n}{2}[120 + 24 - 24n] = \frac{n}{2}[144 - 24n]$$

$$= 12n\left[\frac{144}{24} - n\right] = \underline{12n[6-n]}$$

12a)



cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A$

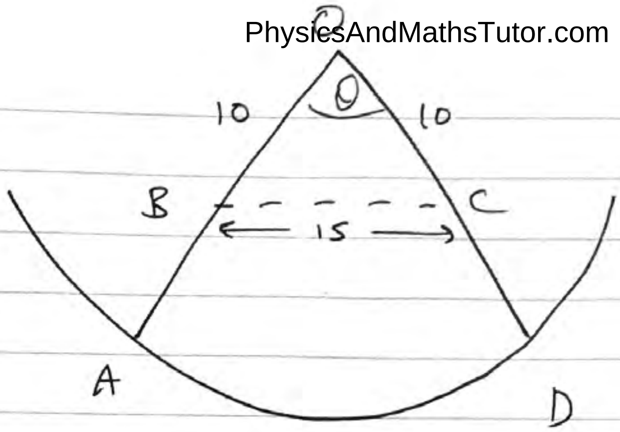
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\therefore \cos\theta = \dfrac{10^2 + 10^2 - 15^2}{2(10)(10)} = -\dfrac{1}{8}$

$\therefore \theta = \cos^{-1}\left(-\frac{1}{8}\right) = \boxed{1.696^c}$

$\theta = 1.696^c$

b)

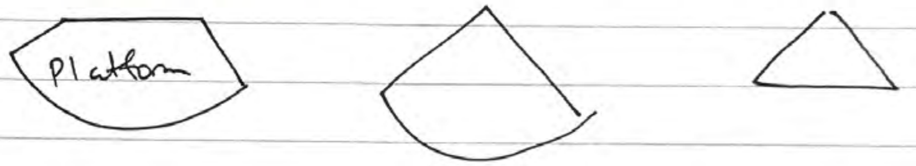


$AB = CD = 22 - 10 = \underline{\underline{12}}\ m.$

AD length $= r\theta = 22 \times 1.696 \simeq 37.31$

$\therefore$ Perimeter $= 12 + 12 + 15 + 37.31 = \boxed{76.3}$

c) Area $ABCD$ = OAD − OBC



$$\text{Area} = \tfrac{1}{2}(22^2)(1.696) - \left(\tfrac{1}{2}\right)(10^2)(\sin(1.696))$$

$$= 410.432 - 49.6$$

$$= \boxed{361 m^2}$$

13a) $y = \dfrac{(x-3)(3x-25)}{x} = \dfrac{3x^2 - 34x + 75}{x}$

$y = 3x - 34 + \dfrac{75}{x} = 3x - 34 + 75x^{-1}$

$\therefore \dfrac{dy}{dx} = 3 - 75x^{-2} = \boxed{3 - \dfrac{75}{x^2}}$

at turning point, $\frac{dy}{dx} = 0$.

$$3 - \frac{75}{x^2} = 0$$

$$3 = \frac{75}{x^2}$$

$$3x^2 = 75 \longrightarrow x^2 = 25$$

$$\therefore x = \sqrt{25} = 5 \quad (x \neq -5,\ x > 0)$$

reject $x = -5$.

at $x = 5$, $y = \frac{(5-3)(3(5)-25)}{5} = -4$

$$\therefore \boxed{(5, -4)}$$

c) $\frac{dy}{dx} = 3 - 75x^{-2}$

$$\frac{d^2y}{dx^2} = 150x^{-3}$$

at $x = 5$, $\frac{d^2y}{dx^2} = \frac{150}{5^3} > 0$

$\therefore$ its a minimum.

d) $P\left(\frac{5}{2}, y\right)$

substituting $x = \frac{5}{2}$ to find y,

$$y = \frac{\left(\frac{5}{2} - 3\right)\left(3\left(\frac{5}{2}\right) - 25\right)}{\frac{5}{2}} = \frac{7}{2} \;//$$

$$\therefore P\left(\frac{5}{2}, \frac{7}{2}\right)$$

$$\frac{dy}{dx} = 3 - \frac{75}{x^2}$$

$\hookrightarrow$ at $x = \frac{5}{2}$, $\frac{dy}{dx} = -9$ // (at P)

$\therefore$ at normal, $m = \frac{1}{9}$ $\quad \left(\frac{1}{9} x - 9 = -1\right)$

$$y - \frac{7}{2} = \frac{1}{9}\left(x - \frac{5}{2}\right)$$

$$y = \frac{1}{9}x - \frac{5}{18} + \frac{7}{2}$$

$$y = \frac{1}{9}x + \frac{29}{9}$$

$$\Rightarrow \boxed{x - 9y + 29 = 0}$$

14a) $x^2 - 2x - 15 = 2x - 3$

$x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

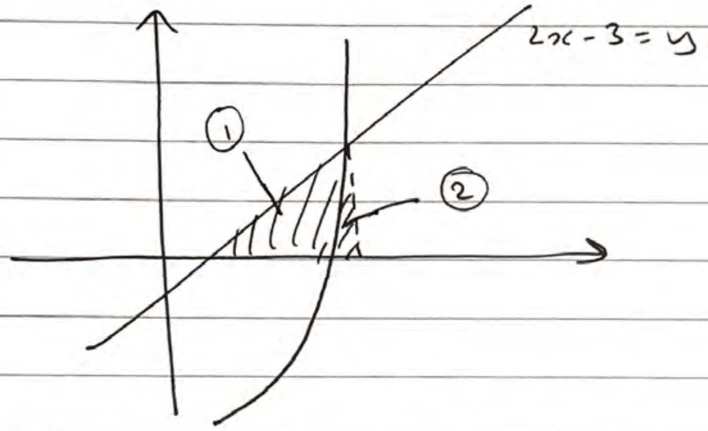
$x = 6, \quad x = -2.$

$y = 2(6) - 3 = 9 \qquad\qquad y = 2(-2) - 3 = -7$

$\therefore$ points are : $A(-2, -7) \quad B(6, 9)$

(A is below x-axis).

b)



line

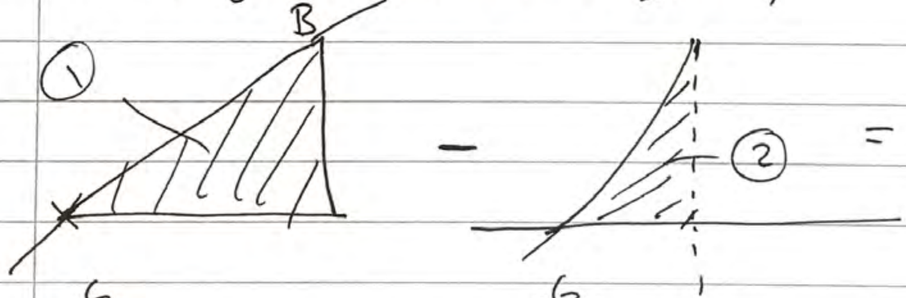
$y = 0, \quad x = \frac{3}{2}$

curve

$y = 0, \quad (x-5)(x+3) = 0$

$x = 5$

To acquire shaded region,



$= R.$

$$= \int_{\frac{3}{2}}^{6} (2x-3)\,dx - \int_{5}^{6} (x^2 - 2x - 15)\,dx = R$$

(OR AREA Δ)

$$R = \left[x^2 - 3x\right]_{\frac{3}{2}}^{6} - \left[\frac{x^3}{3} - x^2 - 15x\right]_{5}^{6}$$

$$= \frac{81}{4} - \frac{13}{3} = \boxed{\frac{191}{12}}$$

15a) $X(0,3)$ $\quad Y(6,11)$

$$m_{xy} = \frac{11-3}{6-0} = \frac{8}{6} = \boxed{\frac{4}{3}}$$

b) $M =$ midpoint of XY.

$$\therefore M\left(\frac{0+6}{2}, \frac{11+3}{2}\right) \Rightarrow M(3,7)$$

Line MZ is perpendicular to XY so gradient will be $-\frac{3}{4}$

$$\left(-\frac{3}{4} \times \frac{4}{3} = -1\right)$$

$$\Rightarrow y - 7 = -\frac{3}{4}(x-3)$$

$$\Rightarrow y = -\frac{3}{4}x + \frac{9}{4} + 7$$

$$\Rightarrow y = -\frac{3}{4}x + \frac{37}{4}$$

$$\Rightarrow \boxed{4y + 3x - 37 = 0}$$

c) using eqn from (b).

$y = 10 \therefore 40 + 3x - 37 = 0$

$3x = -3$

$\boxed{x = -1}$

d) $Z(-1, 10)$ is the centre.

we now need the radius.

radius = ZX (or ZY)

$= \sqrt{(-1-0)^2 + (10-3)^2} = 5\sqrt{2}$.

$\therefore r^2 = 50$

hence eqn of circle: $(x+1)^2 + (y-10)^2 = 50$

expanding: $x^2 + 2x + 1 + y^2 - 20y + 100 = 50$

$x^2 + y^2 + 2x - 20y + 51 = 0$